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SOME PROPERTIES OF SIMPLE ELECTRIC CONDUCTING NETWORKS

by

A. E. Kennelly

UNIVERSITY OF VIRGINIA
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SOME PROPERTIES OF SIMPLE ELECTRIC CONDUCTING NETWORKS.

By A. E. KENNELLY.

(*Read April 26, 1924.*)

It is known that if in any alternating-current network of electric conductors, such as that shown in Fig. 1 (with or without transformers in the meshes), two pairs of terminals are selected, such as a, g and b, h; then, at any given frequency, the system may be regarded as possessing, and as being defined by:

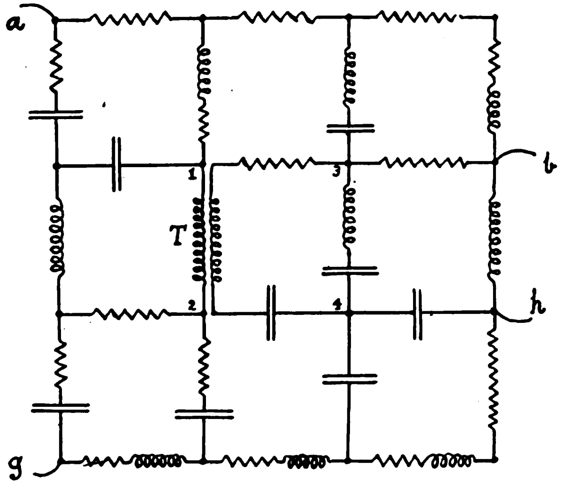

FIG. 1. Diagram representing a network of alternating-current elements.

(1) a hyperbolic angle θ, in general complex (hyperbolic radians or hyps. $\angle$);
(2) a "geomean" surge impedance z_{ab} (ohms $\angle$);
(3) an "inequality ratio" q (numeric $\angle$).

Moreover, the system may be regarded as being reducible, either to the dissymmetrical T of Fig. 2, or to the equivalent dissymmetrical Π of Fig. 3, with respect to the two pairs of selected terminals, and the selected impressed frequency. The numerical values of θ, z_{ab}, and q are obtainable theoretically from three

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measurements of the impedance of the network. In practice, however, it is preferable to make four impedance measurements; viz., two (R_{af} and R_{ag}) at the a, g terminals, with the pair b, h opened (freed), and shorted (grounded) respectively, followed by a similar pair (R_{bf} and R_{bg}) at the b, h terminals, with a and g freed and shorted respectively. The angle θ is then defined by the relation

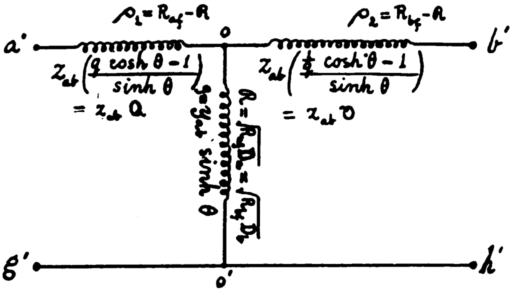

ρ₁ = R_af − R
ρ₂ = R_bf − R
a′
b′
o
z_ab (q cosh θ − 1 / sinh θ) = z_ab Q
z_ab (1/q cosh θ − 1 / sinh θ) = z_ab D
= y_ab sinh θ
R = √(R_ag D_a) = √(R_bg D_b)
g′
o′
h′

FIG. 2. Dissymmetrical T replacing a network with respect to two pairs of terminals.

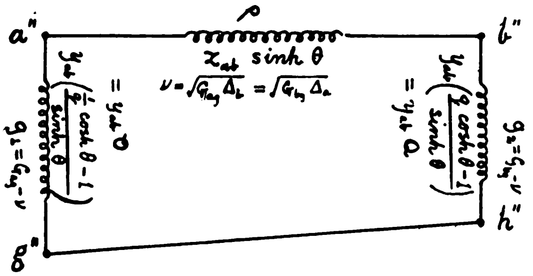

ρ
a″
b″
z_ab sinh θ
ν = √(G_ag Δ_b) = √(G_bg Δ_a)
y_ab (1/q cosh θ − 1 / sinh θ) = y_ab D
y_ab (q cosh θ − 1 / sinh θ) = y_ab Q
g₁ = G_ag − ν
g₂ = G_bg − ν
g″
h″

FIG. 3. Dissymmetrical Π replacing a network with respect to two pairs of terminals.

UV

$$\tanh\theta = \frac{R_{ag}}{R_{af}} = \frac{R_{bg}}{R_{bf}} \qquad \text{numeric } \angle. \quad (1)$$

The apparent surge impedance on the a side becomes $z_{oa} = \sqrt{R_{af}\cdot R_{ag}}$ ohms $\angle$ and that on the b side $z_{ob} = \sqrt{R_{bf}\cdot R_{bg}}$ ohms $\angle$. The geometrical mean of these two gives $z_{ab} = \sqrt{z_{oa}\cdot z_{ob}}$ ohms $\angle$; while their root ratio gives the inequality ratio $q = \sqrt{z_{oa}/z_{ob}}$. From θ, z_{ab} and q, the equivalent T and Π of the network with respect to these particular two pairs of terminals can be found, as in Fig. 3.

It is proposed in this paper to discuss a few elementary cases of networks, from the point of view above outlined.

CASE I. *The terminals b and h are brought indefinitely close to the terminals a and g.* ($\theta = 0$.)

This case is represented in Fig. 4, where the impedances of the network are conventionally shown by simple straight lines. The first pair of terminals, a and g, having been selected, the second pair, b and h, are placed almost in contact with the same. Then the impedances R_{af} and R_{bf} are equal, and may be represented by $\mathfrak{R}$

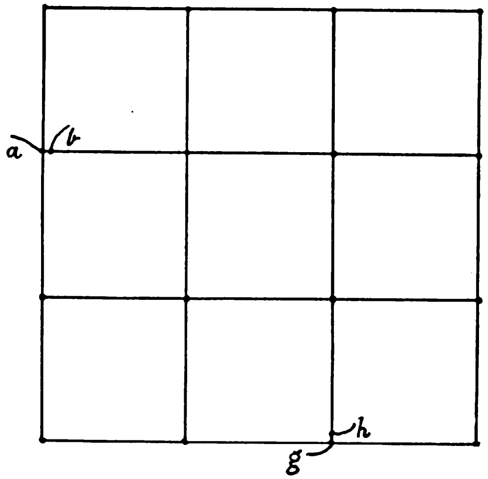
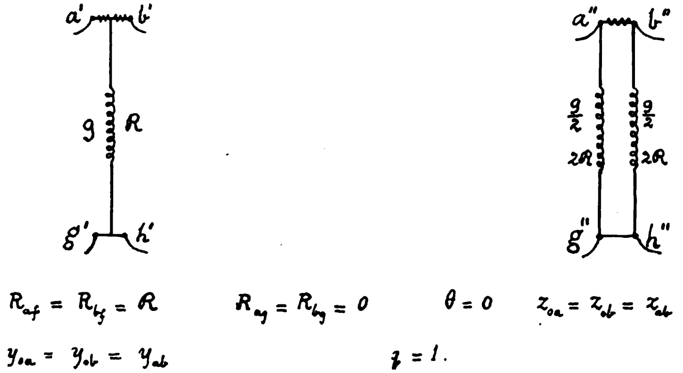


FIG. 4. Case of a network having the two pairs of terminals a,g and b,h brought infinitely close together (θ = 0).



FIGS. 5 and 6. Corresponding equivalent T and Π for Case of Fig. 4.

ohms $\angle$, Fig. 5, the vector reciprocal of which is the admittance g mhos $\angle$. Shorting at either pair of terminals reduces the impedance at the other pair to zero. That is, $R_{ag} = R_{bg} = 0$. Hence $\theta = 0$, or the angle of the network becomes vanishingly small, and $z_{ab} = 0$, or $y_{ab} = \propto$. Also $q = 1$; and there ceases to be inequality. The corresponding equivalents T and Π of the system are shown in Figs. 5 and 6, where the series elements are indefinitely small. The impedance $\mathfrak{R}$ is the impedance offered by the network, at the selected frequency, as measured from either pair of terminals.

CASE II. *The two pairs of terminals are in "conjugate" relation to each other, or are electrically indefinitely remote.* ($\theta = \infty$.)

Fig. 7 represents a particular case of a balanced Wheatstone bridge or quadrilateral. Here the network is reduced to four elements *ab* and *ah*, each of

$$1000 + j2000 = 2236 \angle 63° \, 26' \, 6'' \text{ ohms;}$$

also *bg* and *hg*, each of

$$500 + j1000 = 1118 \angle 63° \, 26' \, 6'' \text{ ohms.}$$

Under such conditions, it is evident that the application of e.m.f. to the terminals *ag* will produce no potential difference between the terminals *bh*, and reciprocally. If we measure impedances between terminals *ag*, we obtain:

$$R_{af} = R_{ag} = 750 + j1500 = 1677.05 \angle 63° \, 26' \, 6'' \text{ ohms } \angle$$

and

$$\sqrt{\frac{R_{ag}}{R_{af}}} = \tanh \theta = 1.$$

$\theta = \infty$, or the angle of the network becomes indefinitely large; while $z_{oa} = 750 + j1500$ ohms. Similarly, measuring impedances between terminals *bh*, we have

$$R_{bf} = R_{bg} = 666.\dot{6} + j1333.\dot{3} = 1490.71 \angle 63° \, 26' \, 6'' \text{ ohms.}$$

Again

$$\sqrt{R_{bg}/R_{bf}} = \tanh \theta = 1.$$

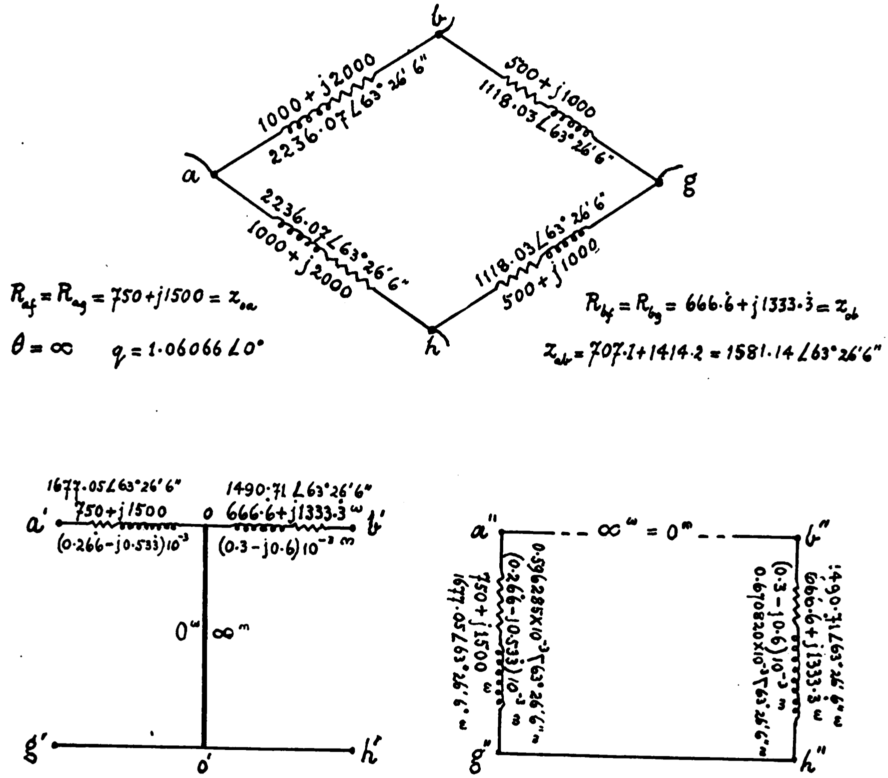
Hence

$$z_{ab} = \sqrt{z_{oa} \cdot z_{ob}} = 707.1 + j1414.2 = 1581.14 \angle 63° \, 26' \, 6'' \text{ ohms}$$

and

$$q = \sqrt{z_{oa}/z_{ob}} = 1.06066 \angle 0^\circ.$$

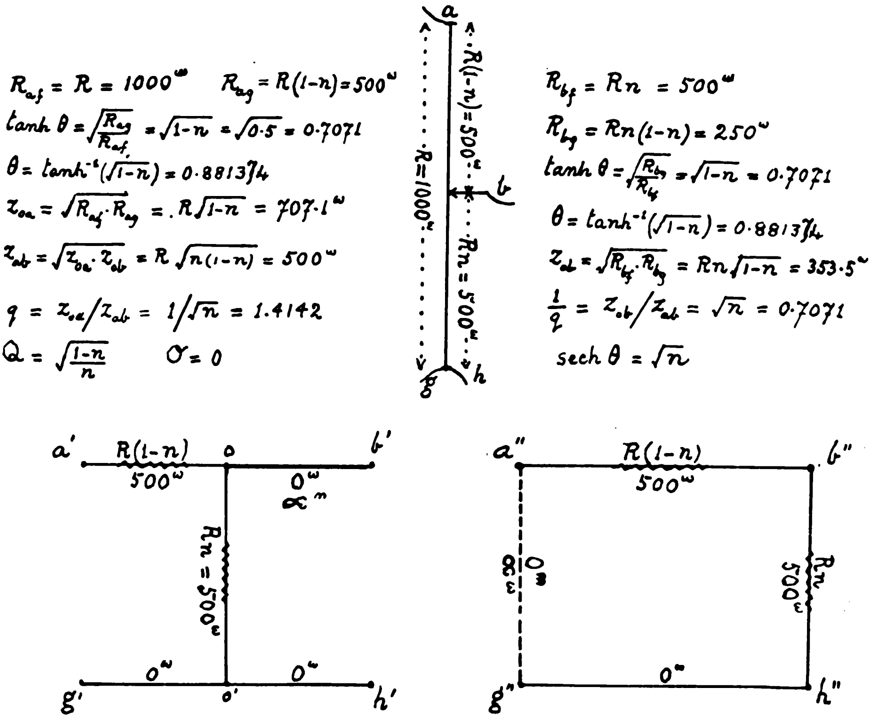
Fig. 8 shows the equivalent dissymmetrical T of the Wheatstone balanced bridge, with respect to the two pairs of terminals. The staff impedance oo' is zero, or the staff admittance is infinity. This infinite leak at oo' prevents any current supplied from one side from affecting conditions at the other. Fig. 9 shows the corresponding dissymmetrical Π. In this case, the architrave impedance $a''b''$ is infinite, or its admittance is zero, which introduces a complete electrical barrier between the two sides of the system.



FIGS. 7, 8 and 9. A Wheatstone Balance, with its equivalent dissymmetrical T and Π. Conjugate Conductor Case ($\theta = \infty$).

CASE III. *The two pairs of terminals are in simple "drop-wire" relation, the position of the contact point being variable at will.* ($\theta = 0$ *to* $\theta = \infty$.)

Fig. 10 presents the simple continuous-current case of a straight uniform wire *ag*, having a total resistance *R*, which is here assumed as 1000 ohms. The terminal *h* is made coincident with *g*; while terminal *b* is a contact, movable along the wire, so as to include *Rn* ohms between *b* and *h*. As shown, $n = 0.5$, or *bh* includes 500 ohms. Then $\tanh \theta = \sqrt{1 - n}$. When $n = 1$, or *b* coincides with *a*, $\tanh \theta = 0$ and $\theta = 0$. Again if $n = 0$, or *b* coincides with *h*, $\tanh \theta = 1$ and $\theta = \infty$. In the case indicated $\theta = 0.88137$ hyp. The geomean surge impedance $z_{ab} = R\sqrt{n(1 - n)}$, in this case 500 ohms. The inequality ratio $q = 1/\sqrt{n} = 1.4142$.


$R_{af} = R = 1000^{\omega}$ $R_{ag} = R(1-n) = 500^{\omega}$
$\tanh\theta = \sqrt{\frac{R_{ag}}{R_{af}}} = \sqrt{1-n} = \sqrt{0.5} = 0.7071$
$\theta = \tanh^{-1}(\sqrt{1-n}) = 0.88137$ hyp
$z_{oa} = \sqrt{R_{af} \cdot R_{ag}} = R\sqrt{1-n} = 707.1^{\omega}$
$z_{ab} = \sqrt{z_{oa} \cdot z_{ob}} = R\sqrt{n(1-n)} = 500^{\omega}$
$q = z_{oa}/z_{ab} = 1/\sqrt{n} = 1.4142$
$Q = \sqrt{\frac{1-n}{n}}$ $\sigma = 0$
a
$R(1-n) = 500^{\omega}$
$R = 1000^{\omega}$
b
$Rn = 500^{\omega}$
g
h
$R_{bf} = Rn = 500^{\omega}$
$R_{bg} = Rn(1-n) = 250^{\omega}$
$\tanh\theta = \sqrt{\frac{R_{bg}}{R_{bf}}} = \sqrt{1-n} = 0.7071$
$\theta = \tanh^{-1}(\sqrt{1-n}) = 0.88137$ hyp
$z_{ob} = \sqrt{R_{bf} \cdot R_{bg}} = Rn\sqrt{1-n} = 353.5^{\omega}$
$\frac{1}{q} = z_{ob}/z_{ab} = \sqrt{n} = 0.7071$
$\operatorname{sech}\theta = \sqrt{n}$
a'
R(1-n)
500^{ω}
o
b'
0^{ω}
∞^{ω}
$Rn = 500^{\omega}$
0^{ω}
0^{ω}
g'
o'
h'
a''
R(1-n)
500^{ω}
b''
∞^{ω}
0^{ω}
500^{ω}
Rn
0^{ω}
g''
h''

FIGS. 10, 11 and 12. Simple Three-Terminal Drop-wire System, with its equivalent dissymmetrical *T* and Π. θ varies from 0 to ∞, as *n* varies from 1 to 0.

For any such drop-wire system, $\theta = 1$ when $n = 0.42$ very nearly; *i.e.*, when the impedance of *bh* is very nearly 42 per cent. of the total impedance *ah*.

SIGNIFICANCE OF THE HYPERBOLIC ANGLE θ OF A NETWORK.

As has already been pointed out, any and every network of electric conductors operated at a single frequency in the steady state, between a pair of driving or sending-end terminals *a*, *g* and a pair of driven or receiving-end terminals *b*, *h*, possesses or subtends a certain complex hyperbolic angle θ. The principal significance of this angle relates to the sending-end and receiving-end impedances of the system, with respect to the two pairs of terminals. Thus the driving-point impedance or sending-end impedance is: with the receiving-end terminals *bh* open,

$$R_{af} = z_{oa} \coth \theta = q z_{ab} \coth \theta, \qquad \text{ohms } \angle, \quad (2)$$

with the same terminals *b*, *h* joined or shorted,

$$R_{ag} = z_{oa} \tanh \theta = q z_{ab} \tanh \theta, \qquad \text{ohms } \angle, \quad (3)$$

and with the same terminals *b*, *h* connected through an impedance σ ohms $\angle$,

$$R_{a\sigma} = z_{oa} \tanh (\theta + \theta') = q z_{ab} \tanh (\theta + \theta'), \qquad \text{ohms } \angle, \quad (4)$$

where

$$\theta' = \tanh^{-1} (\sigma / z_{ob}), \qquad \text{hyps } \angle. \quad (5)$$

Thus, in the simple drop-wire case of Fig. 10, if a load of $\sigma = 200$ ohms is applied across the receiving-end terminals *b*, *h*, the position angle θ' becomes $\tanh^{-1}$ (200/353.55) = 0.64116 hyp., and that of the sending-end terminals $(\theta + \theta')$ becomes 1.52253 hyps. The sending-end impedance $R_{a\sigma}$ is then, by (3), 707.1 × tanh 1.52253 = 642.86 ohms. This result can be obtained more readily, without reference to θ, by applying a terminal load of $\sigma = 200$ ohms to the terminal b' of the equivalent *T* in Fig. 11, or of the equivalent Π in Fig. 12, owing to the fundamental simplicity of the system in Fig. 10; but in the general case of a complicated network, the hyperbolic angle θ has considerable advantage in computation.

Again, the receiving-end impedance of a network loaded with σ ohms $\angle$ at the terminals *b*, *h* is:

$$\begin{aligned} Z_{lb} &= z_{ab} \sinh \theta + q\sigma \cosh \theta, && \text{ohms } \angle, \\ &= q(z_{ob} \sinh \theta + \sigma \cosh \theta), && \text{ohms } \angle. \quad (5a) \end{aligned}$$

If $\sigma = 0$, or the terminals *b*, *h* are shorted,

$$Z_{lb} = z_{ab} \sinh \theta, \qquad \text{ohms } \angle. \quad (6)$$

Thus, with a load of $\sigma = 200$ ohms applied at the receiving-end terminals of Fig. 10, as before, $Z_{lb} = 500 \sinh 0.88137 + 1.4142 \times 200 \cosh 0.88137 = 500 + 400 = 900$ ohms; so that an e.m.f. of 1 volt applied at the terminals *a*, *g* would deliver a current of 1/900 ampere through the load σ, across terminals *b*, *h*. This is readily checked by inspection of Figs. 11 and 12.

CASE IV. *A transformer is included in the network.*

When a transformer is included in the network, as at *T* in Fig. 1, it has long been known[1] that it might be replaced by a certain equivalent conductive relation. So long as the secondary-circuit system is separated from the primary system outside of the transformer, the conversion from inductive to conductive connection is ordinarily easy. When, however, as in Fig. 1, the secondary system is joined to the primary system, or is electrically tested for phase relations in reference to the primary system, some discrepancies present themselves that need to be cleared up, in order to reach a definite and unambiguous result. Some ambiguity is to be found in recent literature on the subject. It seems that only very recently[2] the effects of reversing the connections between primary and secondary windings, on the equivalent *T*, have been pointed out. It is hoped that the following discussion may aid in enabling general agreement to be reached.

An "ideal" two-winding transformer may be defined, for present purposes, as a transformer devoid of power losses and of magnetic leakage. It will, therefore, have vanishingly small resistance in either winding, and no hysteresis in the magnetic circuit. All the magnetic flux links with all the turns. The inductances of the windings being $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ henries, respectively, their mutual inductance will be $\mu = \pm k\sqrt{\boldsymbol{L}_1\boldsymbol{L}_2}$ henries, where k, the coupling coefficient, is unity. The algebraic signs of $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ are essentially positive; but that of μ is open to either positive or negative interpretation. The impedance of the windings at an impressed angular velocity ω radians per second, where $\omega = 2\pi f$, and f is the frequency in cycles per second, will be $z_1 = j\boldsymbol{L}_1\omega$, and $z_2 = j\boldsymbol{L}_2\omega$ ohms, respectively; while the mutual impedance will be $M = \pm j\mu\omega$ ohms, either sign being available. We also assume that, in an ideal transformer, the capacitance, either between the two windings, or between turns in one and the same winding, is negligible in its effects; so that when one winding is supplied with a sinusoidal current of I rms. amperes, the e.m.f. thereby induced in the other winding is $MI = \pm j\mu\omega I$ rms. volts, in quadrature with the current, the sign being taken as either positive or negative, according to the direction of reference adopted.

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[1] Bibliography 2.

[2] Bibliography 18.

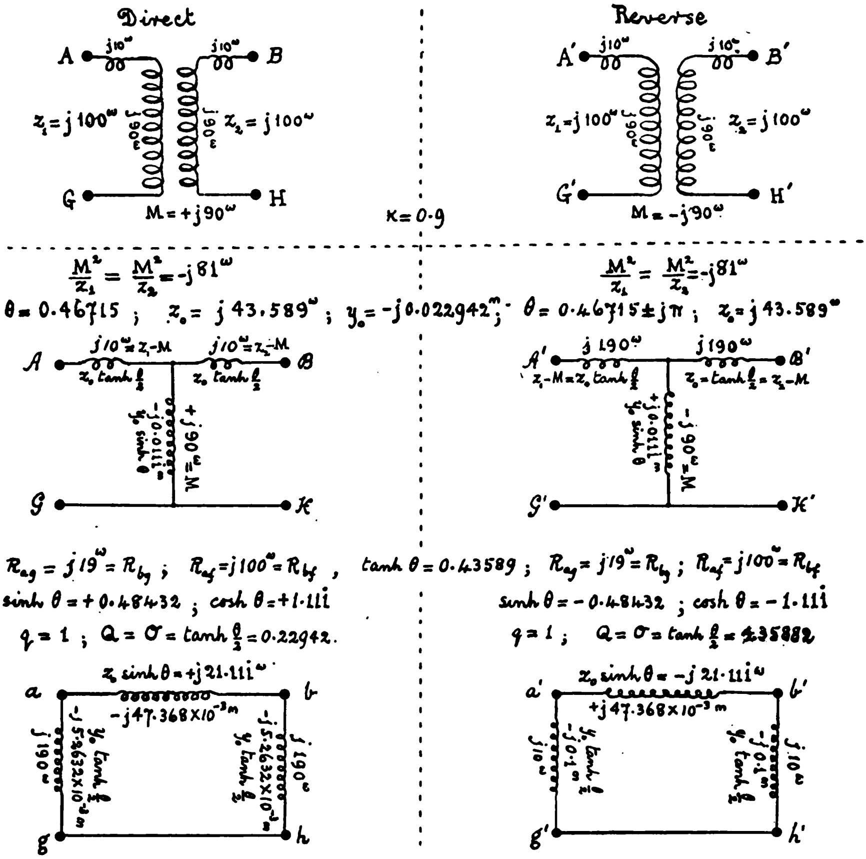

FIG. 13. Pure Level Transformer with 10 per cent. magnetic leakage, direct and reverse connections.

An actual transformer, with its two low-resistance windings suitably laid on together over a wooden toroidal core, may furnish, for many purposes, a satisfactory approximation to an ideal transformer, when operated at a low frequency. When, however, the coupling coefficient k falls materially short of unity, the transformer, otherwise regarded as ideal, may be called a "pure" transformer; *i.e.*, as free from power losses, or capacitance disturbances.

A transformer with equal numbers of primary and secondary turns, or unity ratio of transformation, is sometimes described as a "level transformer." Consequently, a level pure transformer has no power losses but has magnetic leakage. Consequently, $z_1 = jL_1\omega = z_2 = jL_2\omega = jL\omega$ ohms, and $M = \pm j\mu\omega = \pm jkL\omega$.

Fig. 13 indicates the connections of a level pure transformer, with 10 per cent. of magnetic leakage, or a coupling coefficient $k = 0.9$. Each winding has zero resistance, and $j100$ ohms reactance, of which $j90$ ohms is in that part of the winding completely interlinked with the other, with $j10$ ohms in the remaining part which is magnetically dissociated. On the left-hand side marked "direct" the mutual impedance is taken as $M = +j90$ ohms, and on the right-hand side, marked "reverse," as $-j90$ ohms, the second winding being reversed in the two cases with respect to the two pairs of terminals *AG* and *BH*.

With one pair of these terminals open, say *BH*, the impedance offered by the winding connected to the other pair *AG* will be $R_{af} = z_1 = j100$ ohms. Closing the second circuit by shorting terminals *BH* will increase[3] the impedance, as measured at *AG*, by an amount equal to $M^2/z_2 = -j81$ ohms. There is no ambiguity in the sign of this increment, since it depends on M^2. It is the same in either the direct or reverse connection, and in either winding of each. Consequently, in all four cases, $R_{ag} = j100 - j81 = j19$ ohms. Moreover, by symmetry, $q = 1$ for a level transformer, and the geomean surge impedance $\sqrt{R_{af} \cdot R_{ag}}$ becomes $z_o = j43.589$ ohms.

Taking the ratio R_{ag}/R_{af}, or 0.19, as $\tanh^2 \theta$, where θ is the angle of the transformer, we obtain $\tanh \theta = 0.43589$.

Owing to the double sign of M, we find that there are two values of the angle θ, corresponding to the relation $\tanh \theta = 0.43589$. One is $\theta = 0.46715$ hyp., a "real" angle corresponding to the $+$ sign of M, and the other is $\theta = 0.46715 \pm j\pi$ hyp., a complex angle, with a circular component of π radians, or 2 quadrants.

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[3] Bibliography 1.

This complex θ corresponds to the $-$ sign of M. To these two different angles correspond the two different equivalent T's, and the two different equivalent Π's, shown in Fig. 13. In the T's, the staff or shunt impedance is M, with its appropriate sign; while in each T branch is the impedance $z_1 - M$, or $z_2 - M$, to correspond. In the direct-connection T, the branch impedances are $j10$ ohms each. In the reverse-connection T, they are $j190$ ohms each. Moreover, if there were no magnetic leakage ($k = 1$), the direct T branches would have zero impedance, and the reverse T branches would be $2M$, or $j200$ ohms each.

It does not seem possible to discriminate between the two T's or the two Π's, by any regular tests that can be applied to this transformer, in which the secondary circuit is kept distinct from, and independent of, the primary. That is, the sign of M remains ambiguous, so long as the primary and secondary systems are independent. Either the direct or the reverse T may be used, with any primary source on one side, and any impedance applied as a secondary load to the other. In other words, one cannot tell whether the connection is direct or reverse, and the terminals may be reversed, on the primary or on the secondary side, without affecting the impedances, voltages, currents or powers on either side.

If, however, we interconnect the primary and secondary sides of the transformer, either conductively, or through another transformer, the resulting physical conditions remove the ambiguity, and determine the sign of M. Either the direct connection, or the reverse connection, is implied, and one only of the two T's, or of the two Π's, must be used, to represent the behavior of the transformer. To take a simple case, we may connect terminal A to terminal B, and likewise terminal G to terminal H. This will place the two windings in parallel connection. If the connection is direct, an examination of the equivalent T, $\mathcal{A}\mathcal{B}\mathcal{G}\mathcal{H}$, shows that the joint impedance of the two windings will be $\frac{j10}{2} + j90 = j95$ ohms; whereas, if the connection is reverse, the other T, $\mathcal{A}'\mathcal{B}'\mathcal{G}'\mathcal{H}'$, shows the joint impedance to be $\frac{j190}{2} - j90 = j5$ ohms, or 19 times less. In the direct case, the current received by the parallel-connected transformer, from 100-volt a.-c. mains, would be $-j1.053$ amperes; a simple excitation current; while in the reverse case, it would be $-j20$ amperes, which would be a short-circuit current. Moreover, if there were no magnetic leakage ($k = 1$), the parallel-connected level ideal transformer would take 1 ampere in the direct-connection case, and an infinitely great current in the reverse-connection case.

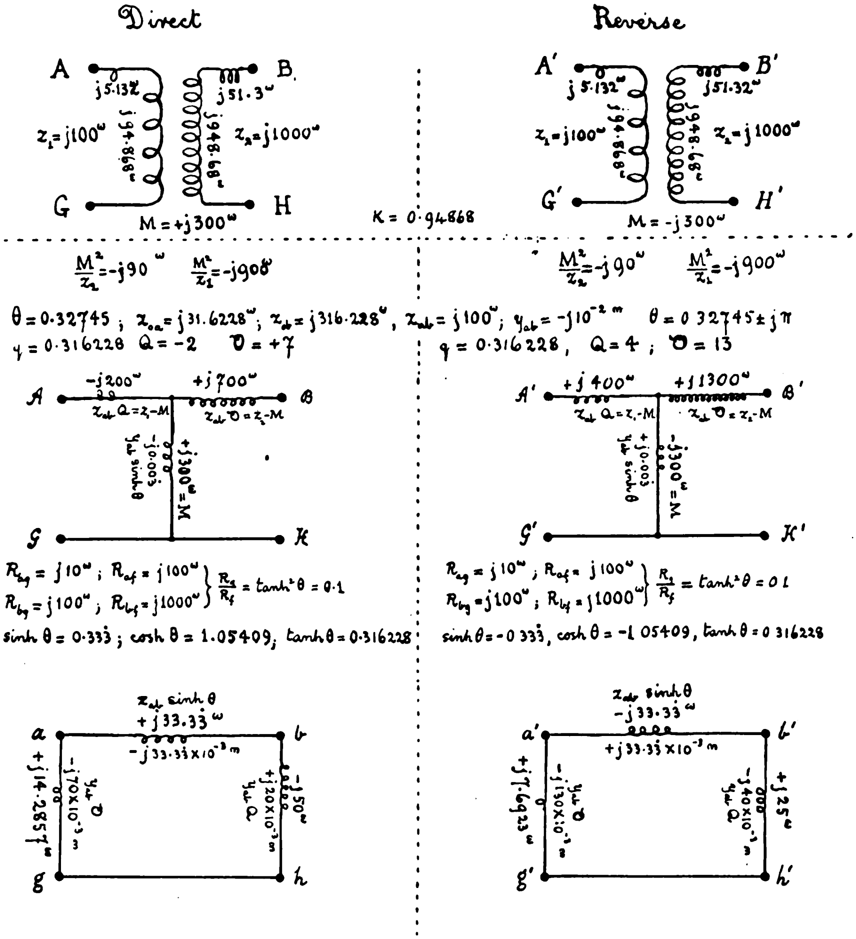

FIG. 14. Pure Unlevel Transformer, direct and reverse connections.

Again, if in Fig. 13 we connect together terminals G and H, so as to put the two windings in series, the impedance measured between the end terminals A and B will be found, from an examination of the equivalent T's, to be $j20$ ohms with direct connection, and $j380$ ohms with reverse connection. The difference between these two measurements ($j360$), as is well known, is four times[4] the mutual reactance M. Similar remarks apply to the cases indicated in Figs. 14 and 15.

Referring to the network of Fig. 1, with the transformer T' embedded in it, the interconnection between its primary and secondary systems dispels the ambiguity as to the sign of the mutual inductance μ and of the mutual impedance M. One or other of the corresponding equivalent T's will be rendered necessary, according as the connection of the transformer is direct or reverse. It might not be necessary to ascertain which connection was actually used. For two given pairs of terminals, such as a, g and b, h, Fig. 1, the values of R_{ag} and R_{af}, R_{bg} and R_{bf} should give the correct final equivalents T and Π of the entire system, transformer included; but in analyzing the composition of the final T and Π, either the direct or the reverse T and Π of the transformer, considered separately, would have to be used.

Consequently, although in dealing with a complicated network, like Fig. 1, it may be neither necessary nor desirable to know which connection is used in the transformer; yet, in order to replace the transformer by an equivalent T in the network, say between terminals 1, 2, 3 and 4, it would become necessary to ascertain, say by a test of the transformer temporarily disconnected from the network, whether the direct T or the reverse T should be employed.

In Fig. 14, the pure transformer indicated is unlevel, and has a transformation ratio of 10, with a coupling coefficient $k = 0.94868$; so that $z_1 = j100$ ohms, $z_2 = j1000$ ohms, and $M = \pm j300$ ohms, with $+j300$ for direct, and $-j300$ for inverse connection. The impedance due to the secondary closed circuit in the primary is $-j90$ ohms on the z_1 side and $-j900$ ohms on the z_2 side. Here $z_{ab} = j100$ ohms, and $q = \sqrt{0.1}$. The value of $\tanh \theta$ being 0.31623, the two values of θ are 0.32745 and $0.32745 + j\pi$ hyps., respectively, according to the $+$ or $-$ sign of the mutual inductance μ.

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[4] Bibliography 12 and 13.

If two transformers, of the type shown in Fig. 14, were operated as step-up transformers, in parallel for 100-volt mains, and with their secondary terminals joined, they would operate in parallel, sharing the exciting current, provided that both were direct connected, or else both reverse connected. If, however, one were direct and the other were reverse, they would act as mutual short circuits to each other. In any of these cases, the distribution of

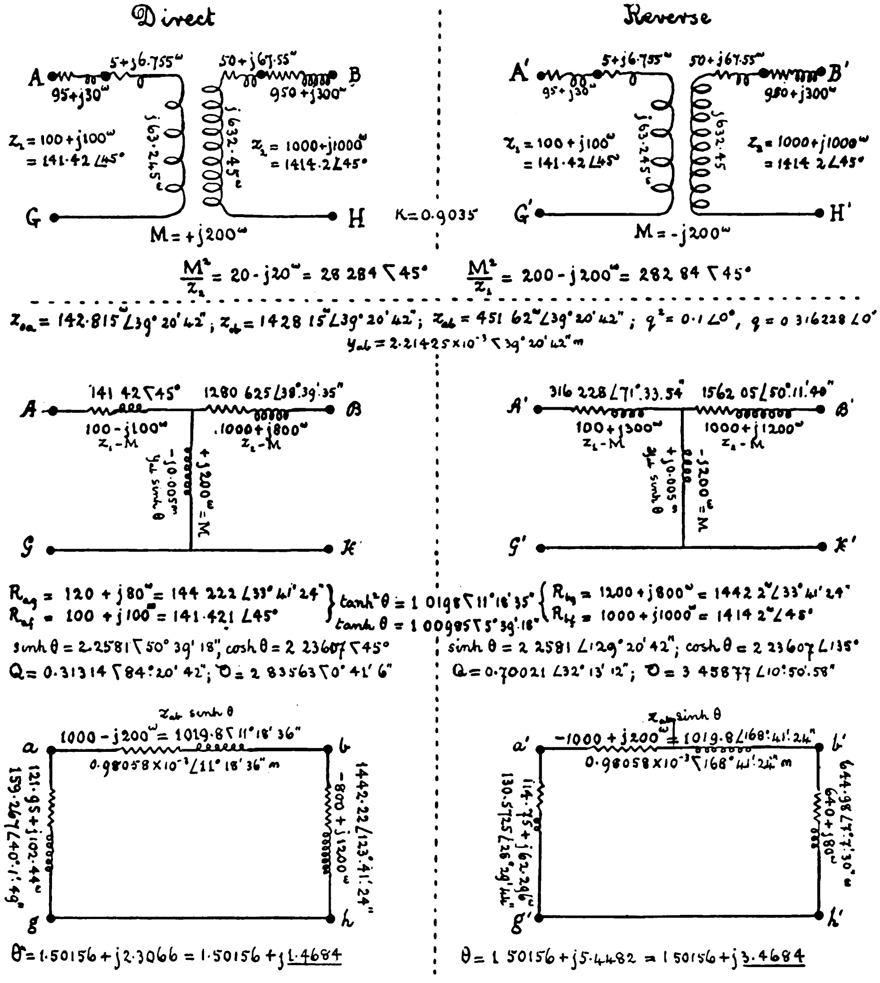
Direct
Reverse
A
5+j6.755ω
95+j30ω
50+j67.55ω
B
950+j300ω
z₁ = 100+j100ω
= 141.42∠45°
j63.245ε
j632.45ε
z₂ = 1000+j1000ω
= 1414.2∠45°
G
H
M = +j200ω
K=0.9035
A'
5+j6.755ω
95+j30ω
50+j67.55ω
950+j300ω
B'
z₁ = 100+j100ω
= 141.42∠45°
j63.245ε
j632.45
z₂ = 1000+j1000ω
= 1414 2∠45°
G'
H'
M = -j200ω
M²/z₂ = 20-j20ω = 28 284∖45°
M²/z₁ = 200-j200ω = 282 84∖45°
z₀ₐ = 142.815ω∠39° 20' 42"; z₀b = 1428 15ω∠39° 20' 42"; zab = 451 62ω∠39° 20' 42"; q² = 0.1∠0°, q = 0 316228∠0°
yab = 2.21425×10⁻³∖39° 20' 42"℧
141 42∖45°
1280 625∠38°.39'.35"
A
B
100-j100ω
z₁-M
.1000+j800ω
z₂-M
-j0.005℧
yab sinh θ
+j200ε = M
G
K
316 228∠71°.33.54"
1562 05∠50°.11'.40"
A'
B'
100+j300ω
z₁-M
1000+j1200ω
z₂-M
+j0.005℧
yab sinh θ
-j200ε = M
G'
K'
Rag = 120 + j80ω = 144 222∠33° 41' 24"
Raf = 100 + j100ω = 141.421∠45°
tanh²θ = 1 0198∖11° 18' 35"
tanh θ = 1 00985∖5° 39'.18"
Rbg = 1200 + j800ω = 1442 2ω∠33° 41' 24"
Rbf = 1000 + j1000ω = 1414 2ω∠45°
sinh θ = 2.2581∖50° 39' 18"; cosh θ = 2 23607∖45°
Q = 0.31314∖84°.20' 42"; 𝔗 = 2 83563∖0° 41' 6"
sinh θ = 2 2581∠129° 20' 42"; cosh θ = 2 23607∠135°
Q = 0.70021∠32° 13' 12"; 𝔗 = 3 45877∠10°.50'.58"
zab sinh θ
1000 - j200ω = 1019.8∖11° 18' 36"
0.98058×10⁻³∠11° 18' 36"℧
a
b
121.95+j102.44ω
159.267∠40°.1'.49"
-800 + j1200ω
1442.22∠123°.41'.24"
g
h
θ = 1.50156+j2.3066 = 1.50156+j1.4684
zab sinh θ
-1000 + j200ω = 1019.8∠168°.41'.24"
0.98058×10⁻³∖168° 41'.24"℧
a'
b'
114.75+j62.296ε
130.5725∠28° 29' 44"
640+j80ω
644.98∠7°.7'.30"ω
g'
h'
θ = 1 50156+j5.4482 = 1 50156+j3.4684

FIG. 15. General Unlevel Transformer with Air Core and External Impedances. Direct and reverse connections.

voltage and current in the system might be found by suitable connection of the T's and Π's of Fig. 14 with an appropriate source.

Fig. 15 represents the case of a general type of transformer, without core losses, but with resistance in the primary and secondary circuits, both internal and external. This case might correspond to an ordinary power transformer of 10 : 1 transformation ratio, if the shunt leak or staff of the T had resistance as well as reactance, to represent the loss of power in the magnetic circuit. The angle of the transformer is complex in this case; $\theta = 1.50156 + j2.3066$ for direct connection, and $1.50156 + j5.4482$ for reverse connection, the two differing by π radians, or 2 quadrants, in the circular component. The angle θ can ordinarily be found by charts to a fair degree of precision.[5]

TERMINOLOGY.

Question may be raised as to the applicability of the terms "direct" and "reverse," here suggested for the two relative connections of primary and secondary terminals. This is a mere matter of terminology, and the choice of these terms has no necessary implication in the arithmetical conditions under consideration. Whatever terms may be selected, they should not involve any disparagement of either mode of connection, because for some purposes it may be desirable to use the one, and for other purposes, the other.

A connection is "direct," in the sense here proposed, when, if the transformer were ideal and level, the branch impedances of the equivalent T would be zero or the terminal A would be in *direct* connection with terminal B, as against $2M$ ohms in each branch when reverse. The angle θ of the transformer would be real for direct, as against complex for reverse, connection. The staff impedance M of the equivalent T would be $+ j\mu\omega$ direct, as against $- j\mu\omega$ reverse. The two windings would be applied in the same sense and towards the same parts direct, as against relatively crossed or with reversed wires when reverse. When the primary and secondary windings are connected in series, their total impedance is greater when connected reverse than when connected direct, by the amount $4M$.

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[5] Bibliography 10.

CONCLUSIONS.

1. The angle θ of a network, with respect to two pairs of terminals, may vary from 0, when the terminals are close together (a near b and g near h), to infinity, when they are very remote, or stand in conjugate relation. In a simple drop-wire circuit, the full-range movement of the contact point will cause θ to vary from 0 to ∞.

2. Any transformer may be regarded as possessing an angle θ, with respect to its primary and secondary terminals. This angle differs between the direct and reverse connection by $j\pi$ radians or $j2$ quadrants.

3. The algebraic sign of μ, the mutual inductance of any induction coil or transformer, is ambiguous. The ambiguity cannot ordinarily be cleared up, and ordinarily does not need to be, so long as the primary and secondary circuits are separated, either sign being admissible. When, however, the primary and secondary circuits are connected either conductively, or inductively through a second transformer, the ambiguity disappears, and the sign is either + or − according as the transformer connection is direct or reverse. In an ideal transformer, the + sign involves a value of $M = + j\mu\omega$ for the staff impedance of the equivalent T in direct connection and $M = - j\mu\omega$ in reverse connection.

4. Reversing either the primary or the secondary terminals of a transformer, with respect to the corresponding winding, will change the connection from direct to reverse, or oppositely.

5. Either the direct or inverse equivalent T will represent the electrical behavior of a transformer with separated primary and secondary circuits. Only one of them, corresponding to the actual internal connections, can, however, properly apply, when the primary and secondary circuits are brought into connection.

6. The direct or reverse connection of a transformer or transformers, embedded in a conducting network, may not be made evident by tests at two pairs of terminals on the network, but in order to replace a transformer in the network by the proper equivalent T, its connections must be ascertained, as to whether they are direct or reverse.

7. As a matter of terminology, the validity of the terms "direct"

and "reverse," as above suggested and defined, is debatable; but some such distinguishing terms would be useful.

LIST OF SYMBOLS EMPLOYED.

$D = R_f - R_g$, difference in impedances due to shorting at opposite ends (ohms $\angle$).

$\Delta = G_g - G_f$, difference in admittances due to shorting at opposite ends (mhos $\angle$).

f, frequency of sinusoidal current in a circuit (cycles per second).

G_{af}, G_{ag}, admittance of a network at a terminals when freed and shorted at b respectively (mhos $\angle$).

g, admittance of the staff leak in an equivalent T (mhos $\angle$).

g_1, g_2, admittances of the pillar leaks in an equivalent Π (mhos $\angle$).

θ, angle of a transformer or of a network with respect to two pairs of terminals (hyperbolic radians or hyps. $\angle$).

θ', angle of a load at the receiving end of a network (hyps. $\angle$).

I, value of a sinusoidal alternating current (amperes).

$j = \sqrt{-1}$.

$k = \dfrac{\mu}{\sqrt{L_1 L_2}}$, coupling coefficient of a transformer (numeric).

L_1, L_2, inductances of the two windings of a transformer (henries).

$M = \pm j\mu\omega$, mutual impedance of a transformer (ohms $\angle$).

$\mu = \pm k\sqrt{L_1 L_2}$, mutual inductance of two transformer windings (henries).

n, the ratio of secondary to primary impedance in a simple drop-wire system (numeric $\angle$).

ν, architrave admittance of a Π (mhos $\angle$).

Π, type of equivalent circuit having one impedance in series and two terminal leaks.

$\pi = 3.14159\ldots$

$Q = \dfrac{q\cosh\theta - 1}{\sinh\theta}$, a factor in assigning the elements of an equivalent T or Π (numeric $\angle$).

$\partial = \dfrac{\frac{1}{q}\cosh\theta - 1}{\sinh\theta}$, a factor in assigning the elements of an equivalent T or Π (numeric $\angle$).

$q = \sqrt{z_{oa}/z_{ob}}$, inequality factor of a network or of its equivalent T or Π (numeric $\angle$).

$\mathfrak{R}$, impedance of the staff leak in an equivalent T (ohms $\angle$).

R_{af}, impedance of a network measured from the a, g terminals with the b, h terminals free (ohms $\angle$).

R_{ag}, impedance of a network measured from the a, g terminals with the b, h terminals shorted (ohms $\angle$).

$R_{a\sigma}$, impedance of a network measured from the a, g terminals with the b, h terminals connected through an impedance of σ ohms.

ρ, architrave impedance of an equivalent Π (ohms $\angle$).

ρ_1, ρ_2, impedances of the arms or line branches of an equivalent T (ohms $\angle$).

σ, a terminal load impedance at the receiving-end terminals (ohms $\angle$).

$y_{ab} = 1/z_{ab}$, geomean surge admittance of a network or of its equivalent T or Π (mhos $\angle$).

Z_{lb}, receiving-end impedance of a network at b terminals (ohms $\angle$).

z_1, z_2, impedances of the two windings of a transformer (ohms $\angle$).

z_o, surge impedance of a symmetrical network (ohms $\angle$).

$z_{oa} = \sqrt{R_{ag} \cdot R_{af}}$, surge impedance of a network from the a, g terminals (ohms $\angle$).

$z_{ob} = \sqrt{R_{bg} \cdot R_{bf}}$, surge impedance of a network from the b, h terminals (ohms $\angle$).

$z_{ab} = \sqrt{z_{oa} \cdot z_{ob}}$, geomean surge impedance of a network or of its equivalent T or Π (ohms $\angle$).

$\omega = 2\pi f$, angular velocity impressed on a network or transformer (radians/second).

BIBLIOGRAPHY.

Without pretensions as to completeness.

1. "Impedance of Mutually Inductive Circuits," A. E. Kennelly, *The Electrician*, London, Vol. XXXI., Oct. 27, 1893, pp. 699–700.
2. "Alternating-Current Phenomena," C. P. Steinmetz, New York, 1897, p. 184.
3. "Leerlauf und Kurzschluss," J. L. La Cour, 1904.
4. "Artificial Lines for Continuous Currents in the Steady State," A. E. Kennelly, *Proc. Am. Ac. Arts and Sc.*, Vol. XLIV., 1904, No. 4, p. 97.
5. "Equivalent Circuits of Composite Lines in the Steady State," A. E. Kennelly, *Proc. Am. Ac. Arts and Sc.*, Vol. XLV., 1909, No. 3, p. 31.

6. "Leçons sur l'Electricité," Eric Gerard, Paris, 8th edition, 1910, Vol. 2, p. 31.
7. "Cisoidal Oscillations," G. A. Campbell, *Trans. A. L. E. E.*, Vol. XXX., Part 2, April, 1911, p. 873.
8. "The Application of Hyperbolic Functions to Electrical Engineering Problems," A. E. Kennelly, London and New York, 1912, p. 158.
9. "Theory and Calculation of Electric Circuits," La Cour and Bragstad, Translation, 1913.
10. "Chart Atlas of Complex Hyperbolic and Circular Functions," A. E. Kennelly, Cambridge, Mass., 1913.
11. "Alternating Currents," C. E. Magnusson, New York, 1916, p. 144.
12. "The Principles of Electric Wave Telegraphy and Telephony," J. A. Fleming, London, 3d Edition, 1916, p. 169.
13. "Electrical Measurements," F. A. Laws, New York, 1917, p. 416.
14. "Alternating-Current Planevector Potentiometer Measurements at Telephonic Frequencies," A. E. Kennelly and Edy Velander, *Proc. Am. Phil. Soc.*, Vol. LVIII., April, 1919, p. 97.
15. "Electric Oscillations and Electric Waves," G. W. Pierce, New York, 1920.
16. "Lecture Notes on Telephony," lectures at Harvard University, Jan., 1922, K. S. Johnson.
17. "Dissymmetrical Electrical Conducting Networks," A. E. Kennelly, *Journal A. I. E. E.*, Feby., 1923, Vol. XLII., No. 2, pp. 112–122.
18. "Telephone Transformers," W. L. Casper, *Journal A. I. E. E.*, Mar., 1924, pp. 197–209.

HARVARD UNIVERSITY AND MASS. INST. TECHNOLOGY,
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LIST OF RESEARCH BULLETINS — Continued

Number

16. Magnetic Flux Distribution in Annular Steel Laminæ. A. E. Kennelly and P. L. Alger; March, 1918.

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*20. Alternating-Current Planevector Potentiometer Measurements at Telephonic Frequencies. A. E. Kennelly and Edy Velander; July, 1919.

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28. A Simple Harmonic Analyser; V. Bush. Electromechanical Device for Rapid Schedule Harmonic Analysis of Complex Waves; Frederick S. Dellenbaugh, Jr. Alignment Chart for Circular and Hyperbolic Functions of a Complex Argument in Rectangular Coördinates; V. Bush. January, 1922.

29. A Mechanical Frequency-Meter of Telephonic Range. A. E. Kennelly and C. L. Manneback; January, 1922.

30. An Integral Equation for Skin Effect in Parallel Conductors. C. L. Manneback; May, 1922.

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*Out of Print.

LIST OF RESEARCH BULLETINS — Continued

Number

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45. Some Properties of Simple Electric Conducting Networks. A. E. Kennelly; December, 1924.